Online Supplement to "Realized Drift"

Sébastien Laurent[†] Roberto Renò[‡] Shuping Shi[§]

April 15, 2022

This document is an Online Supplement containing additional results and proofs for Laurent, Renò, and Shi (2022), titled "Realized Drift". Section B provides the limiting distributions of RV, RiceV, and RAC in border cases. Section C shows the limiting properties of RAC^o, that is preaveraged realized autocovariance.

B Additional mathematical results: border cases

Corollary B.1. *Under Assumption 2.1, as $n \to \infty$, the limiting distribution of* $\mathsf{RV} - \mathsf{IV}$ *in border cases is as follows.*

(1-2) When $\beta = 1/4$ and $\alpha < 3/4$:

$$\left(\frac{n}{\log(n)}\right)^{1/2} [\mathsf{RV} - \mathsf{IV}] \xrightarrow{d} \mathcal{N}\left(0, 2\zeta'^{\sigma^4}_{(2,0)}\right).$$

(1-3) When $\beta < 1/4$ and $\alpha = 3/4$:

$$n^{1/2} [\mathsf{RV} - \mathsf{IV}] \xrightarrow{d} \mathcal{N}\left(\zeta^{\mu^2}_{(\frac{3}{4},2,0)}, 2\int_0^1 \sigma_s^4 (1-s)^{-4\beta} ds\right).$$

(2-4) When $\beta \geqslant 1/4$ and $\alpha - \beta = 1/2$:

$$n^{1-2\beta} [\mathsf{RV} - \mathsf{IV}] \xrightarrow{d} \mathcal{N}\left(\zeta^{\mu^2}_{(\frac{1}{2}+\beta,2,0)}, 2\zeta^{\sigma^4}_{(2\beta,2,0)} + 4\zeta^{\mu^2\sigma^2}_{(\frac{1}{2}+\beta,2,0),(2\beta,1,0)}\right).$$

[†]Aix-Marseille University, CNRS & EHESS, Aix-Marseille Graduate School of Management – IAE, e-mail: sebastien.laurent@univ-amu.fr

[‡]University of Verona, Dipartimento di Scienze Economiche, e-mail: roberto.reno@univr.it

[§]Macquarie University, Department of Economics, e-mail: shuping.shi@mq.edu.au

(3-4) When $\beta < 1/4$ and $\alpha + \beta = 1$:

$$n^{1/2}\left[\mathsf{RV} - \mathsf{IV} - \Delta_n^{2\beta}\zeta^{\mu^2}_{(1-\beta,2,0)}\right] \xrightarrow{d} \mathcal{N}\left(0, 2\int_0^1 \sigma_s^4(1-s)^{-4\beta}ds + 4\zeta^{\mu^2\sigma^2}_{(1-\beta,2,0),(2\beta,1,0)}\right).$$

The above convergences are stable in law.

Corollary B.2. *Under Assumption 2.1, as $n \to \infty$, the limiting distribution of $\mathsf{RiceV}(k) - \mathsf{IV}$ in border cases is as follows.*

(1-2) When $\beta = 1/4$ and $\alpha < 3/4$:

$$\left(\frac{n}{\log(n)}\right)^{1/2}[\mathsf{RiceV}(k) - \mathsf{IV}] \xrightarrow{d} \mathcal{N}\left(0, 2\zeta'^{\sigma^4}_{(2,0)} + \zeta'^{\sigma^4}_{(1,k),(1,0)}\right).$$

(1-3) When $\beta < 1/4$ and $\alpha = 3/4$:

$$n^{1/2}[\mathsf{RV} - \mathsf{IV}] \xrightarrow{d} \mathcal{N}\left(\zeta^{\mu^2}_{(\frac{3}{4},2,0)} - \zeta^{\mu^2}_{(\frac{3}{4},1,0),(\frac{3}{4},1,k)}, 3\int_0^1 \sigma_s^4(1-s)^{-4\beta}ds\right).$$

(2-4) When $\beta \geqslant 1/4$ and $\alpha - \beta = 1/2$:

$$\begin{aligned}&n^{1-2\beta}[\mathsf{RV} - \mathsf{IV}]\\ &\xrightarrow{d} \mathcal{N}\left(\zeta^{\mu^2}_{(\frac{1}{2}+\beta,2,0)} - \zeta^{\mu^2}_{(\frac{1}{2}+\beta,1,0),(\frac{1}{2}+\beta,1,k)}, 2\zeta^{\sigma^4}_{(2\beta,2,0)} + \zeta^{\sigma^4}_{(2\beta,1,k),(2\beta,1,0)} + V_4^{\mathsf{RiceV}}\left(\frac{1}{2}+\beta,\beta\right)\right).\end{aligned}$$

(3-4) When $\beta < 1/4$ and $\alpha + \beta = 1$:

$$\begin{aligned}&n^{1/2}\left[\mathsf{RV} - \mathsf{IV} - \Delta_n^{2\beta}\left(\zeta^{\mu^2}_{(1-\beta,2,0)} - \zeta^{\mu^2}_{(1-\beta,1,0),(1-\beta,1,k)}\right)\right]\\ &\xrightarrow{d} \mathcal{N}\left(0, 3\int_0^1 \sigma_s^4(1-s)^{-4\beta}ds + V_4^{\mathsf{RiceV}}(1-\beta,\beta)\right).\end{aligned}$$

The above convergences are stable in law.

Corollary B.3. *Under Assumption 2.1, as $n \to \infty$, the limiting distribution of $\mathsf{RAC}(k)$ in border cases is as follows.*

(1-2) When $\beta = 1/4$ and $\alpha < 3/4$:

$$\left(\frac{n}{\log(n)}\right)^{1/2}\mathsf{RAC}(k) \xrightarrow{d} \mathcal{N}\left(0, \zeta'^{\sigma^4}_{(1,k),(1,0)}\right).$$

(1-3) When $\beta < 1/4$ and $\alpha = 3/4$:

$$n^{1/2}\,\mathsf{RAC}(k) \xrightarrow{d} \mathcal{N}\left(\zeta^{\mu^2}_{(\frac{3}{4},1,0),(\frac{3}{4},1,k)}, \int_0^1 \sigma_s^4 (1-s)^{-4\beta} ds\right).$$

(2-4) When $\beta \geqslant 1/4$ and $\alpha - \beta = 1/2$:

$$n^{1-2\beta}\,\mathsf{RAC}(k) \xrightarrow{d} \mathcal{N}\left(\zeta^{\mu^2}_{(\frac{1}{2}+\beta,1,0),(\frac{1}{2}+\beta,1,k)}, \zeta^{\sigma^4}_{(2\beta,1,k),(2\beta,1,0)} + V_4^{\mathsf{RAC}}\left(\frac{1}{2}+\beta,\beta\right)\right).$$

(3-4) When $\beta < 1/4$ and $\alpha + \beta = 1$:

$$n^{1/2}\left[\mathsf{RAC}(k) - \Delta_n^{2\beta}\zeta^{\mu^2}_{(1-\beta,1,0),(1-\beta,1,k)}\right] \xrightarrow{d} \mathcal{N}\left(0, \int_0^1 \sigma_s^4 (1-s)^{-4\beta} ds + V_4^{\mathsf{RAC}}(1-\beta,\beta)\right).$$

The above convergences are stable in law.

C Additional mathematical results: RAC^o

Lemma C.1. *Let $A_s = \mu_s (1-s)^{-\alpha}$. The weight function $g(.)$ satisfies conditions in Assumption A.1. Then,*

$$(1)\ \sum_{j=1}^{l_n} g\left(\frac{j}{l_n}\right)\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n} A_s ds - \int_{i\Delta_n}^{(i+l_n)\Delta_n} g\left(\frac{s-i\Delta_n}{l_n\Delta_n}\right) A_s ds = O_p\left(\Delta_n^{1-\alpha}\right),$$

$$(2) \int_{i\Delta_n}^{(i+l_n)\Delta_n} g\left(\frac{s-i\Delta_n}{l_n\Delta_n}\right) A_s ds = O_p\left(\Delta_n^{1-\alpha} l_n\right).$$

Proof. (1) Write

$$\int_{i\Delta_n}^{(i+l_n)\Delta_n} g\left(\frac{(s-i\Delta_n)/\Delta_n}{l_n}\right) A_s ds = \sum_{j=1}^{l_n}\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n} g\left(\frac{(s-i\Delta_n)/\Delta_n}{l_n}\right) A_s ds.$$

Let

$$\alpha_n = \sum_{j=1}^{l_n}\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n}\left[g\left(\frac{j}{l_n}\right) - g\left(\frac{(s-i\Delta_n)/\Delta_n}{l_n}\right)\right] A_s ds.$$

By the mean value theorem, together with the boundness of g' and μ_s, we have, for each interval $[(i+j-1)\Delta_n, (i+j)\Delta_n]$ there exists a $\xi \in \left(\frac{s-i\Delta_n}{\Delta_n}, j\right)$ such that:

$$
\begin{aligned}
|\alpha_n| &\leqslant \frac{1}{l_n}\sum_{j=1}^{l_n}\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n}\left|g'\left(\frac{\xi_i}{l_n}\right)(j-\frac{s-i\Delta_n}{\Delta_n})\right||A_s|\,ds \\
&\leqslant C\frac{1}{l_n}\int_{i\Delta_n}^{(i+l_n)\Delta_n}|A_s|ds \\
&\leqslant C\frac{1}{l_n}\int_{i\Delta_n}^{(i+l_n)\Delta_n}(1-s)^{-\alpha}ds = C\Delta_n^{1-\alpha}(n-i)^{-\alpha} = O_p\left(\Delta_n^{1-\alpha}\right)
\end{aligned}
$$

since, by the first order Taylor series expansion,

$$
\int_{i\Delta_n}^{(i+l_n)\Delta_n}(1-s)^{-\alpha}ds = \frac{\Delta_n^{1-\alpha}}{1-\alpha}\left[(n-i)^{1-\alpha}-(n-i-l_n)^{1-\alpha}\right] \approx \Delta_n^{1-\alpha}l_n(n-i)^{-\alpha}.
$$

(2) By the boundness of μ_s and $g(.)$,

$$
\begin{aligned}
\int_{i\Delta_n}^{(i+l_n)\Delta_n} g\left(\frac{s-i\Delta_n}{l_n\Delta_n}\right)A_s ds &= \int_{i\Delta_n}^{(i+l_n)\Delta_n} g\left(\frac{s-i\Delta_n}{l_n\Delta_n}\right)\mu_s(1-s)^{-\alpha}ds \\
&\leqslant C\int_{i\Delta_n}^{(i+l_n)\Delta_n}(1-s)^{-\alpha}ds \approx C\Delta_n^{1-\alpha}l_n(n-i)^{-\alpha} = O_p\left(\Delta_n^{1-\alpha}l_n\right).
\end{aligned}
$$

□

Proof of Theorem 3.5. We first derive the first order property of the signal component in (16). By definition,

$$
\begin{aligned}
\bar{r}_{i\Delta_n} &= -\sum_{j=0}^{l_n-1}h_j^n X_{(i+j)\Delta_n} = \sum_{j=1}^{l_n}g\left(\frac{j}{l_n}\right)\left(X_{(i+j)\Delta_n}-X_{(i+j-1)\Delta_n}\right) \\
&= \sum_{j=1}^{l_n}g\left(\frac{j}{l_n}\right)\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n}A_s ds + \sum_{j=1}^{l_n}g\left(\frac{j}{l_n}\right)\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n}B_s dW_s,
\end{aligned}
$$

where $B_s = \sigma_s(1-s)^{-\beta}$. Since $k_n > l_n$, the conditional expectation of $\bar{r}_{i\Delta_n}\bar{r}_{(i-k_n)\Delta_n}$ is

$$
\begin{aligned}
& E\left[\bar{r}_{i\Delta_n}\bar{r}_{(i-k_n)\Delta_n}|\mathcal{F}_{i-k_n}^n\right] \\
=& \left(\sum_{j=1}^{l_n}g\left(\frac{j}{l_n}\right)\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n}A_s ds\right)\left(\sum_{j=1}^{l_n}g\left(\frac{j}{l_n}\right)\int_{(i-k_n+j-1)\Delta_n}^{(i-k_n+j)\Delta_n}A_s ds\right) \\
=& \left(\int_{i\Delta_n}^{(i+l_n)\Delta_n}g\left(\frac{s-i\Delta_n}{l_n\Delta_n}\right)A_s ds\right)\left(\int_{(i-k_n)\Delta_n}^{(i-k_n+l_n)\Delta_n}g\left(\frac{s-i\Delta_n}{l_n\Delta_n}\right)A_s ds\right)[1+o_p(1)] \\
=& O_p\left(\Delta_n^{2(1-\alpha)}l_n^2\right).
\end{aligned}
$$

The third and fourth lines come from Lemma C.1.

Next, we investigate the first order property of the three terms brought by the noise in (16). Let

$$\begin{aligned} Q_{i,k_n} =& \omega^2 \sum_{j_1=0}^{l_n-1} h_{j_1}^n \varepsilon_{(i-k_n+j_1)\Delta_n} \sum_{j_2=0}^{l_n-1} h_{j_2}^n \varepsilon_{(i+j_2)\Delta_n} \\ & - \omega \bar{r}_{i\Delta_n} \sum_{j=0}^{l_n-1} h_j^n \varepsilon_{(i-k_n+j)\Delta_n} - \omega \bar{r}_{(i-k_n)\Delta_n} \sum_{j=0}^{l_n-1} h_j^n \varepsilon_{(i+j)\Delta_n}. \end{aligned}$$

Under Assumption 3.4, $|\gamma_s| \leqslant \frac{K}{s^v}$ with $v > 1$. The conditional expectation

$$E\left[Q_{i,k_n}|\mathcal{F}_{i-k_n}^n\right] = \omega^2 E\left[\sum_{j_1=0}^{l_n-1} h_{j_1}^n \varepsilon_{(i-k_n+j_1)\Delta_n} \sum_{j_2=0}^{l_n-1} h_{j_2}^n \varepsilon_{(i+j_2)\Delta_n} |\mathcal{F}_{i-k_n}^n\right]$$

$$\begin{aligned} &= \omega^2 \sum_{s=k_n-l_n+1}^{k_n} \sum_{j=0}^{s-k_n+l_n-1} h_j^n h_{j+k_n-s}^n E\left(\varepsilon_{i+j}\varepsilon_{i+j-s}|\mathcal{F}_{i-k_n}^n\right) \\ &+ \omega^2 \sum_{s=k_n+1}^{k_n+l_n} \sum_{j=s-k_n}^{l_n-1} h_j^n h_{j+k_n-s}^n E\left(\varepsilon_{i+j}\varepsilon_{i+j-s}|\mathcal{F}_{i-k_n}^n\right) \\ &= \omega^2 \sum_{s=k_n-l_n+1}^{k_n} \gamma_s \sum_{j=0}^{s-k_n+l_n-1} h_j^n h_{j+k_n-s}^n + \omega^2 \sum_{s=k_n+1}^{k_n+l_n} \gamma_s \sum_{j=s-k_n}^{l_n-1} h_j^n h_{j+k_n-s}^n \\ &\leqslant \frac{C}{l_n^2} \sum_{s=k_n-l_n+1}^{k_n} \gamma_s \left(s-k_n+l_n\right) + \frac{C}{l_n^2} \sum_{s=k_n+1}^{k_n+l_n} \gamma_s \left(l_n - s + k_n\right) \\ &= \frac{C}{l_n^2} \sum_{m=1}^{l_n} \left(\gamma_{m+k_n-l_n} + \gamma_{k_n+l_n-m}\right) m \leqslant \frac{2CK}{l_n^2} \sum_{m=1}^{l_n} \frac{m}{\left(m+k_n-l_n\right)^v} \end{aligned}$$

since $h_j^n = g\left(\frac{j+1}{l_n}\right) - g\left(\frac{j}{l_n}\right) = g\left(\xi_j\right)' \frac{1}{l_n} \leqslant \frac{C}{l_n}$ with $\frac{j}{l_n} \leqslant \xi_j \leqslant \frac{j+1}{l_n}$, and

$$\begin{aligned} \left|\gamma_{m+k_n-l_n} + \gamma_{k_n+l_n-m}\right| &\leqslant \left|\gamma_{m+k_n-l_n}\right| + \left|\gamma_{k_n+l_n-m}\right| \\ &\leqslant \frac{K}{\left(m+k_n-l_n\right)^v} + \frac{K}{\left(k_n+l_n-m\right)^v} \leqslant \frac{2K}{\left(m+k_n-l_n\right)^v} \end{aligned}$$

from Assumption 3.4 and the fact that $m + k_n - l_n \leqslant k_n + l_n - m$. Furthermore, since $\frac{m}{(m+k_n-l_n)^v} < 1$, we have

$$E\left[Q_{i,k_n}|\mathcal{F}_{i-k_n}^n\right] < \frac{2CK}{l_n} = O_p\left(\frac{1}{l_n}\right).$$

Therefore, the signal component dominates the noise terms and

$$\mathsf{RAC}^{o}\left(k_n\right) = \sum_{i=k_n+1}^{n} \bar{r}_{i\Delta_n}\bar{r}_{(i-k_n)\Delta_n} + o_p(1) \tag{1}$$

when

$$\Delta_n^{2(1-\alpha)} l_n^3 \to \infty.$$

□