

VALUE-AT-RISK FOR LONG AND SHORT TRADING POSITIONS

Pierre Giot¹ and Sébastien Laurent²

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Abstract

In this paper we model Value-at-Risk (VaR) for daily asset returns using a collection of parametric univariate and multivariate models of the ARCH class based on the skewed Student distribution. We show that models that rely on a symmetric density distribution for the error term underperform with respect to skewed density models when the left and right tails of the distribution of returns must be modelled. Thus, VaR for traders having both long *and short* positions is not adequately modelled using usual normal or Student distributions. We suggest using an APARCH model based on the skewed Student distribution (combined with a time-varying correlation in the multivariate case) to fully take into account the fat left and right tails of the returns distribution. This allows for an adequate modelling of large returns defined on long and short trading positions. The performances of the univariate models are assessed on daily data for three international stock indexes and three U.S. stocks of the Dow Jones index. In a second application, we consider a portfolio of three U.S. stocks and model its long and short VaR using a multivariate skewed Student density.

Keywords: Value-at-Risk, (Multivariate) skewed Student distribution, APARCH, Short trading, Expected short-fall

JEL classification: C52, C53, G15

¹Corresponding author: Department of Business Administration & CEREFIM, University of Namur, Rempart de la Vierge 8, B-5000 Namur, Belgium, Tel.: +3281724887 and Center for Operations Research and Econometrics, Université catholique de Louvain, Belgium; email: pierre.giot@fundp.ac.be

²CREPP, Université de Liège, Center for Operations Research and Econometrics, Université catholique de Louvain, and Department of Quantitative Economics, Maastricht University; email: S.Laurent@ulg.ac.be

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1 Introduction

In recent years, the tremendous growth of trading activity and the well-publicized trading loss of well known financial institutions (see Jorion, 2000, for a brief history of these events) has led financial regulators and supervisory committee of banks to favor quantitative techniques which appraise the possible loss that these institutions can incur. Value-at-Risk has become one of the most sought-after techniques as it provides a simple answer to the following question: with a given probability (say α), what is my predicted financial loss over a given time horizon? The answer is the VaR at level α , which gives an amount in the currency of the traded assets (in dollar terms for example) and is thus easily understandable.

It turns out that the VaR has a simple statistical definition: the VaR at level α for a sample of returns is defined as the corresponding empirical quantile at $\alpha\%$. Because of the definition of the quantile, we have that, with probability $1 - \alpha$, the returns will be larger than the VaR. In other words, with probability $1 - \alpha$, the losses will be smaller than the dollar amount given by the VaR.¹ From an empirical point of view, the computation of the VaR for a collection of returns thus requires the computation of the empirical quantile at level α of the distribution of the returns of the portfolio.

Most models in the literature focus on the computation of the VaR for negative returns (see van den Goorbergh and Vlaar, 1999 or Jorion, 2000). Indeed, it is assumed that traders or portfolio managers have long trading positions, i.e. they bought the traded asset and are concerned when the price of the asset falls. In this paper we focus on modelling VaR for portfolios defined on long *and short* trading positions. Thus we model VaR for traders having either bought the asset (long position) or short-sold it (short position).² In the first case, the risk comes from a drop in the price of the asset, while the trader loses money when the price increases in the second case (because he would have to buy back the asset at a higher price than the one he got when he sold it). Correspondingly, one focuses in the first case on the left side of the distribution of returns, and on the right side of the distribution in the second case.

Because the distribution of asset (stocks and stock indexes in this paper) returns is often not symmetric (see Section 2.2), we show that ‘usual’ parametric VaR models of the RiskMetrics and ARCH class have a tough job in modelling correctly the left and right tails of the distribution of returns. This is also true for the so-called asymmetric GARCH models where the asymmetry refers to the relationship between the conditional variance and the lagged squared error term.³

¹Contrary to some wide-spread beliefs, the VaR does not specify the maximum amount that can be lost.

²An asset is short-sold by a trader when it is first borrowed and subsequently sold on the market. By doing this, the trader hopes that the price will fall, so that he can then buy the asset at a lower price and give it back to the lender. See Sharpe, Alexander and Bailey (1999) for general information on trading procedures.

³Indeed, as pointed out by El Babsiri and Zakoian (1999), although such asymmetric GARCH models allow positive and negative changes to have different impacts on future volatilities, the two components of the innovation

To alleviate these problems, we first introduce an univariate skewed Student Asymmetric Power ARCH (APARCH) model (Ding, Granger and Engle, 1993) to model the VaR for portfolios defined on long (long VaR) and short (short VaR) trading positions. We compare the performance of this new model with the ones of the RiskMetrics, normal and Student APARCH models and show that the new model brings about considerable improvements in correctly forecasting one-day-ahead VaR for long and short trading positions on daily stock indexes (English FTSE, U.S. NASDAQ, Japanese NIKKEI) and daily U.S. stocks (Alcoa, MacDonald and Merck). For the skewed Student APARCH model, we also compute the expected short-fall and the average multiple of tail event to risk measure as these two measures supplement the information given by the empirical failure rates.⁴

Secondly we tackle the long and short VaR issue in a multivariate framework. Thus instead of focusing on univariate returns, we introduce a multivariate APARCH model which features time-varying correlations (between the chosen assets) and which relies on the multivariate skewed Student distribution to provide asymmetry in the long and short VaR forecasts. This model thus takes into account the salient empirical features of a portfolio of daily stock returns: volatility clustering with the APARCH structure, time-varying correlations, fat-tails and skewness. While volatility clustering, fat-tails and skewness were also ‘univariate features’, the multivariate analysis adds the additional challenge of the time-varying correlations. This last issue is important, see for example Longin and Solnik (1995) who show that markets become more closely related during periods of high volatility (i.e when accurate VaR forecasts are most needed), or more recently Brooks and Persaud (2000) for an application of VaR during market crashes.⁵ The multivariate model is applied to a portfolio of three U.S. stocks and the long and short VaR performance for the portfolio is assessed in-sample and out-of-sample.

Note that Mittnik and Paolella (2000) have recently introduced an APARCH model combined with an asymmetric generalized Student distribution to model VaR for negative returns. While the analysis in their paper is sometimes similar to ours, there are some significant differences. First, we focus on the joint behavior of VaR models for long *and short* trading positions, i.e. we look at both how large negative and positive returns are taken into account by the model (Mittnik have - up to a constant - the same volatilities, while it is desirable to allow an asymmetric confidence interval around the predicted volatility in the VaR application.

⁴While we focus exclusively on parametric models, other approaches are possible, such as Danielsson and de Vries (2000) who combine an historical simulation method (i.e. non parametric technique) for the interior of the distribution of returns with a fitted distribution based on extreme value theory for the most extreme returns. In this setting, normal and extreme events are thus modelled using two different methods. With the skewed Student APARCH model we aim to model left and right tail VaRs with a single parametric method for a wide range of values for α .

⁵See also Billio and Pelizzon (2000) for an application of a multivariate switching regime model to the computation of the VaR of a portfolio of stocks.

and Paoletta, 2000, focus on long VaR only). Secondly, our empirical analysis deals with daily data for stock indexes, in contrast to exchange rate data for the other paper. That usual datasets such as the daily returns for European and U.S. indexes indicate the need for these types of models is an important issue, as most studies usually focus on ‘exotic’ series for justifying the use of these models. Thirdly, we assess the performances of the models by computing Kupiec (1995)’s LR tests on the empirical failure rates. For the new model, we also compute the expected short-fall and the average multiple of tail event to risk measure. Last, from a methodological point of view, following Lambert and Laurent (2001) we re-express the estimated parameters in terms of the mean and variance of the skewed Student distribution (instead of the mode and the dispersion) and consider its multivariate extension to forecast the VaR of a given portfolio.

As indicated in Christoffersen and Diebold (2000), volatility forecastability (such as featured by ARCH class models) decays quickly with the time horizon of the forecasts. An immediate consequence is that volatility forecastability is relevant for short time horizons (such as daily trading), but not for long time horizons on which portfolio managers usually focus. In this paper, we are consistent with these characteristics of volatility forecastability as we focus on daily returns and analyze VaR performance for daily trading portfolios made up of long and short positions.

The rest of the paper is organized in the following way. In Section 2, we describe the symmetric and asymmetric univariate VaR models and we assess their performance with a dataset of daily stock index and stock returns. The multivariate model is introduced in Section 3 where it is applied to a portfolio of three stocks. Section 4 concludes.

2 Univariate VaR models

In this section we present the univariate VaR models that are used to model the long and short sides of daily trading positions. We successively consider the RiskMetrics, normal APARCH, Student APARCH and skewed Student APARCH models. Prior to the description of the chosen models, we first characterize our datasets and highlight the salient features of the daily returns for three international stock indexes and three U.S. stocks of the Dow Jones index.

In the first empirical application we consider daily return data for a collection of three stock market indexes and three U.S stocks (Source: Yahoo Finance): the English FTSE stock index (FTSE, 04/01/1988 - 21/12/2000), the U.S. NASDAQ (NASDAQ, 11/10/1984 - 21/12/2000) and the Japanese NIKKEI stock indexes (NIKKEI, 4/1/1984 - 21/12/2000); the Alcoa stock (AA, 03/01/1990 - 03/05/2002), the MacDonald stock (MCD, 03/01/1990 - 03/05/2002) and Merck stock (MRK, 03/01/1990 - 03/05/2002), where the numbers in parentheses are the start and end dates for our sample and the first symbol inside the parentheses is the short notation (or ticker for the U.S. stocks) for the series that are used in the tables and comments below.

For all price series p_t , daily returns (in %) are defined as $y_t = 100 [\ln(p_t) - \ln(p_{t-1})]$. Descriptive characteristics for the returns series are given in Table 1. While the time spans for the six financial assets are different, the six returns series display similar statistical properties as far as the third and fourth moments are concerned. More specifically, the returns series are skewed (either negatively or positively) and the large returns (either positive or negative) lead to a large degree of kurtosis. The Ljung-Box Q-statistics of order 10 on the squared series indicate a high serial correlation in the second moment or variance.

Descriptive graphs (level of index, daily returns, density of the daily returns vs. normal and QQ-plots against the normal distribution) for each index are given in Figures 1-6. The density graphs and the QQ-plot against the normal distribution show that all returns distributions exhibit fat tails. Moreover, the QQ-plots indicate that fat tails are not symmetric.

These figures indicate that the six time series exhibit volatility clustering as periods of low volatility mingle with periods of high volatility and large positive and negative returns. Autoregressive Conditional Heteroscedasticity (ARCH) models (Engle, 1982) are now routinely used to describe and forecast volatility clustering in financial time series. Since the seminal paper by Engle in 1982, numerous extensions have been put forward, see Engle (1995), Bera and Higgins (1993) or Palm (1996), but they all share the same goal which is the modelling of the conditional variance as a function of past (squared) returns and associated characteristics. Because quantiles are direct functions of the variance in parametric models, ARCH class models immediately translate into conditional VaR models. As mentioned in the introduction, these conditional VaR models are important for characterizing short term risk regarding intradaily and daily trading positions.

In the next two sub-sections we characterize the four selected volatility models - the RiskMetrics, normal, Student and skewed Student APARCH models - and we compute and fully characterize the corresponding VaR results for long and short trading positions. Note that we thus use three symmetric (RiskMetrics, normal and student APARCH) and one asymmetric (skewed student APARCH) models.⁶

2.1 VaR models

To characterize the models, we consider a collection of daily returns, y_t , with $t = 1 \dots T$. Because daily returns are known to exhibit some serial autocorrelation⁷, we fit an AR(n) structure on the

⁶We stress that, by symmetric and asymmetric models, we mean a possible asymmetry in the distribution of the error term (i.e. whether it is skewed or not), and not the asymmetry in the relationship between the conditional variance and the lagged squared innovations (the APARCH model features this kind of ‘conditional’ asymmetry whatever the chosen error term).

⁷The serial autocorrelation found in daily returns is not necessarily at odds with the efficient market hypothesis. See Campbell, Lo and MacKinlay (1997) for a detailed discussion.

y_t series for all specifications:

$$\Phi(L)(y_t - \mu) + \varepsilon_t, \quad (1)$$

where $\Phi(L) = 1 - \phi_1 L - \dots - \phi_n L^n$ is an AR lag polynomial of order n . Accordingly, the conditional mean of y_t , i.e. μ_t , is equal to $\mu + \sum_{j=1}^n \phi_j (y_{t-j} - \mu)$. We now consider several specifications for the conditional variance of ε_t .

RiskMetrics

In its most simple form, it can be shown that the basic RiskMetrics model is equivalent to a normal Integrated GARCH (IGARCH) model where the autoregressive parameter is set at a pre-specified value λ and the coefficient of ε_{t-1}^2 is equal to $1 - \lambda$. In the RiskMetrics specification for daily data, $\lambda = 0.94$ and we then have:

$$\varepsilon_t = \sigma_t z_t \quad (2)$$

where z_t is IID $N(0,1)$ and σ_t^2 is defined as:

$$\sigma_t^2 = (1 - \lambda)\varepsilon_{t-1}^2 + \lambda\sigma_{t-1}^2 \quad (3)$$

As indicated in the introduction, the long side of the daily VaR is defined as the VaR level for traders having long positions in the relevant equity index: this is the ‘usual’ VaR where traders incur losses when negative returns are observed. Correspondingly, the short side of the daily VaR is the VaR level for traders having short positions, i.e. traders who incur losses when stock prices increase. How good a model is at predicting long VaR is thus related to its ability to model large negative returns, while its performance regarding the short side of the VaR is based on its ability to take into account large positive returns.

For the RiskMetrics model, the one-step-ahead VaR computed in $t-1$ for long trading positions is given by $\mu_t + z_\alpha \sigma_t$, for short trading positions it is equal to $\mu_t + z_{1-\alpha} \sigma_t$, with z_α being the left quantile at $\alpha\%$ for the normal distribution and $z_{1-\alpha}$ is the right quantile at $\alpha\%$.⁸

Normal APARCH

The normal APARCH (Ding, Granger and Engle, 1993) is an extension of the GARCH model of Bollerslev (1986). It is a very flexible ARCH-type model as it nests at least seven GARCH specifications. The APARCH(1, 1) is:

$$\sigma_t^\delta = \omega + \alpha_1 (|\varepsilon_{t-1}| - \alpha_n \varepsilon_{t-1})^\delta + \beta_1 \sigma_{t-1}^\delta \quad (4)$$

where $\omega, \alpha_1, \alpha_n, \beta_1$ and δ are additional parameters to be estimated. δ ($\delta > 0$) plays the role of a Box-Cox transformation of σ_t , while α_n ($-1 < \alpha_n < 1$), reflects the so-called leverage effect. A positive (resp. negative) value of α_n means that past negative (resp. positive) shocks have a

⁸Note that when computing the VaR, μ_t and σ_t are evaluated by replacing the unknown parameters in Equation (1) by their maximum likelihood estimates (MLE).

deeper impact on current conditional volatility than past positive shocks (see Black, 1976; French, Schwert and Stambaugh, 1987; Pagan and Schwert, 1990). The properties of the APARCH model have been studied recently by He and Teräsvirta (1999a, 1999b).

For the normal APARCH model, the one-step-ahead VaR is computed as for the RiskMetrics model but for the computation of the conditional standard deviation σ_t which is now given by Equation (4) (evaluated at its MLE).

Student APARCH

Previous empirical studies on VaR have shown that models based on the normal distribution usually cannot fully take into account the ‘fat tails’ of the returns distribution. To alleviate this problem, the Student APARCH (or ST APARCH) is introduced:

$$\varepsilon_t = \sigma_t z_t \tag{5}$$

where z_t is IID $t(0, 1, v)$ and σ_t is defined as in Equation (4).

For the Student APARCH model, the VaR for long and short positions is given by $\mu_t + st_{\alpha, v} \sigma_t$ and $\mu_t + st_{1-\alpha, v} \sigma_t$, with $st_{\alpha, v}$ being the left quantile at $\alpha\%$ for the (standardized) Student distribution with (estimated) number of degrees of freedom v and $st_{1-\alpha, v}$ is the right quantile at $\alpha\%$ for this same distribution. Note that because $z_\alpha = -z_{1-\alpha}$ for the normal distribution and $st_{\alpha, v} = -st_{1-\alpha, v}$ for the Student distribution, the forecasted long and short VaR will be equal in both cases.

Skewed Student APARCH

To account for the excess skewness and kurtosis, Fernández and Steel (1998) propose to extend the Student distribution by adding a skewness parameter.⁹ Their procedure allows the introduction of skewness in any continuous unimodal and symmetric (about 0) distribution $g(\cdot)$ by changing the scale at each side of the mode. The main drawback of this density is that it is expressed in terms of the mode and the dispersion. In order to keep in the ARCH ‘tradition’, Lambert and Laurent (2001) re-expressed the skewed Student density in terms of the mean and the variance, i.e. re-parameterize this density in such a way that the innovation process has zero mean and unit variance. Otherwise, it will be difficult to separate the fluctuations in the mean and variance from the fluctuations in the shape of the conditional density (Hansen, 1994).

The innovation process z is said to be (standardized) skewed Student distributed if:

$$f(z|\xi, v) = \begin{cases} \frac{2}{\xi + \frac{1}{\xi}} sg[\xi(sz + m)|v] & \text{if } z < -\frac{m}{s} \\ \frac{2}{\xi + \frac{1}{\xi}} sg[(sz + m)/\xi|v] & \text{if } z \geq -\frac{m}{s} \end{cases} \tag{6}$$

⁹Other (but very similar) asymmetric Student densities have been proposed by Hansen (1994) and Paoletta (1997).

where $g(\cdot|v)$ is the symmetric (unit variance) Student density and ξ is the asymmetry coefficient.¹⁰ m and s^2 are respectively the mean and the variance of the non-standardized skewed Student:

$$m = \frac{\Gamma\left(\frac{v-1}{2}\right)\sqrt{v-2}}{\sqrt{\pi}\Gamma\left(\frac{v}{2}\right)}\left(\xi - \frac{1}{\xi}\right) \quad (7)$$

and

$$s^2 = \left(\xi^2 + \frac{1}{\xi^2} - 1\right) - m^2 \quad (8)$$

Notice also that the density $f(z_t|1/\xi, v)$ is the “mirror” of $f(z_t|\xi, v)$ with respect to the (zero) mean, i.e. $f(z_t|1/\xi, v) = f(-z_t|\xi, v)$. Therefore, the sign of $\log \xi$ indicates the direction of the skewness: the third moment is positive (negative), and the density is skew to the right (left), if $\log \xi > 0$ (< 0).

Lambert and Laurent (2000) show that the quantile function $skst_{\alpha,v,\xi}^*$ of a non standardized skewed student density is:

$$skst_{\alpha,v,\xi}^* = \begin{cases} \frac{1}{\xi}st_{\alpha,v}\left[\frac{\alpha}{2}(1+\xi^2)\right] & \text{if } \alpha < \frac{1}{1+\xi^2} \\ -\xi st_{\alpha,v}\left[\frac{1-\alpha}{2}(1+\xi^{-2})\right] & \text{if } \alpha \geq \frac{1}{1+\xi^2} \end{cases} \quad (9)$$

where $st_{\alpha,v}$ is the quantile function of the (unit variance) Student-t density. It is straightforward to obtain the quantile function of the standardized skewed Student: $skst_{\alpha,v,\xi} = \frac{skst_{\alpha,v,\xi}^* - m}{s}$.

Following Ding, Granger and Engle (1993) and Paoella (1997), if it exists, a stationary solution of Equation (4) is given by:

$$E(\sigma_t^\delta) = \frac{\omega}{1 - \alpha_1 E(|z| - \alpha_n z)^\delta - \beta_1} \quad (10)$$

which depends on the density of z . Such a solution exists if $V = \alpha_1 E(|z| - \alpha_n z)^\delta + \beta_1 < 1$. Ding et al. (1993) derived the expression for $E(|z| - \alpha_n z)^\delta$ in the Gaussian case. Paoella (1997) did the same thing for various non standardized densities. It is straightforward to show that for the standardized skewed Student:¹¹

$$E(|z| - \gamma z)^\delta = \left\{ \xi^{-(1+\delta)}(1+\gamma)^\delta + \xi^{1+\delta}(1-\gamma)^\delta \right\} \frac{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{v-\delta}{2}\right)(v-2)^{\frac{1+\delta}{2}}}{\left(\xi + \frac{1}{\xi}\right)\sqrt{(v-2)}\pi\Gamma\left(\frac{v}{2}\right)} \quad (11)$$

For the skewed Student APARCH model, the VaR for long and short positions is given by $\mu_t + skst_{\alpha,v,\xi}\sigma_t$ and $\mu_t + skst_{1-\alpha,v,\xi}\sigma_t$, with $skst_{\alpha,v,\xi}$ being the left quantile at $\alpha\%$ for the skewed Student distribution with v degrees of freedom and asymmetry coefficient ξ ; $skst_{1-\alpha,v,\xi}$ is the corresponding right quantile. If $\log(\xi)$ is smaller than zero (or $\xi < 1$), $|skst_{\alpha,v,\xi}| > |skst_{1-\alpha,v,\xi}|$ and the VaR for long trading positions will be larger (for the same conditional variance) than the VaR for short trading positions. When $\log(\xi)$ is positive, we have the opposite result.

¹⁰The asymmetry coefficient $\xi > 0$ is defined such that the ratio of probability masses above and below the mean is $\frac{\Pr(z \geq 0|\xi)}{\Pr(z < 0|\xi)} = \xi^2$

¹¹Notice that setting $\xi = 1$ leads to the stationarity condition of the symmetric Student density (with unit variance).

2.2 Empirical application

2.2.1 Estimating the models

In order to perform the VaR analysis in Section 2.2.2, the RiskMetrics, normal APARCH, Student APARCH and skewed Student APARCH are estimated in this section. We do not report full estimation results of the normal and Student APARCH models as they are quite similar to what has been previously shown in the literature. Furthermore, these specifications are encompassed by the skewed Student APARCH model which we fully detail below. The RiskMetrics model does not require any estimation for the conditional volatility specification as it is tantamount to an IGARCH model with predefined values.

Table 2 presents the results for the (approximate maximum likelihood) estimation of the skewed Student APARCH model on the six time series.¹² All computations were performed with G@RCH 2.2, an Ox package with a friendly dialog-oriented interface designed for the estimation and forecast of various univariate ARCH-type models (see Laurent and Peters, 2002).¹³ The data together with the programs used in the paper can be obtained by contacting the second author or downloaded from the *Journal of Applied Econometrics Data Archive* web site.

The skewed Student APARCH model is particularly successful in taking into account the heteroskedasticity exhibited by the data as the Ljung-Box Q-statistic computed on the squared standardized residuals is not significant (except for the MacDonald stock).¹⁴ The three stock market indexes and three U.S. stocks feature relatively similar volatility specifications:

- the autoregressive effect in the volatility specification is strong as β_1 is around 0.9, suggesting a strong memory effects. Indeed, $\alpha_1 E(|z| - \alpha_n z)^\delta + \beta_1$ is just below 1 for five of the six time series and equals 1 for the NASDAQ (indicating that σ_t^δ may be integrated).
- α_n is positive and significant for all datasets, indicating a leverage effect for negative returns in the *conditional* variance specification;
- $\log(\xi)$ is negative and significant for the three stock market indexes, and positive and significant for the three stocks, which implies that the *asymmetry in the Student distribution* is needed to fully model the distribution of returns. Likelihood ratio tests (not reported) also clearly favor the skewed Student density.
- δ is between 1.052 and 1.793 and mostly significantly different from 2. For five of the six series

¹²An AR(2) is sufficient to correct the serial correlation in the conditional mean. Note that to save space, the estimated mean parameters are not reported.

¹³G@RCH 2.2, or a more recent version, can be downloaded for free from the web site <http://www.egss.ulg.ac.be/garch/>.

¹⁴For NASDAQ data, the decrease in the $Q^2(10)$ is impressive as it goes down from more than 3,000 to about 17.

where δ is not significantly different from 1, these results suggest that, instead of modelling the conditional variance (GARCH), it is more relevant to model the conditional standard deviation. This result is in line with those of Taylor (1986), Schwert (1990) and Ding et al. (1993) who indicate that there is substantially more correlation between absolute returns than squared returns, a stylized fact of high frequency financial returns (often called ‘long memory’).

To summarize, these results indicate the need for a model featuring a negative leverage effect (conditional asymmetry) for the conditional variance combined with an asymmetric distribution for the underlying error term (unconditional asymmetry). The skewed Student APARCH model delivers such specifications and we study in Section 2.2.2 if this model improves on symmetric GARCH models when the VaR for long and short returns is needed.

2.2.2 In-sample VaR computation

In this section, we use the estimation results of Section 2.2.1 to compute the one-step-ahead VaR for all models. As financial returns are known to exhibit fat tails (this was confirmed in the descriptive properties of the data given in Table 1), we expect poor performance by the models based on the normal distribution.

All models are tested with a VaR level α which ranges from 5% to 0.25% and their performance is then assessed by computing the failure rate for the returns y_t . By definition, the failure rate is the number of times returns exceed (in absolute value) the forecasted VaR. If the VaR model is correctly specified, the failure rate should be equal to the pre-specified VaR level. In our empirical application, we define a failure rate f_l for the long trading positions, which is equal to the percentage of negative returns smaller than one-step-ahead VaR for long positions. Correspondingly, we define f_s as the failure rate for short trading positions as the percentage of positive returns larger than the one-step-ahead VaR for short positions.

Because the computation of the empirical failure rate defines a sequence of yes/no observations, it is possible to test $H_0 : f = \alpha$ against $H_1 : f \neq \alpha$, where f is the failure rate (estimated by \hat{f} , the empirical failure rate).¹⁵ At the 5% level and if T yes/no observations are available, a confidence interval for \hat{f} is given by $\left[\hat{f} - 1.96\sqrt{\hat{f}(1-\hat{f})/T}, \hat{f} + 1.96\sqrt{\hat{f}(1-\hat{f})/T} \right]$. In this paper these tests are successively applied to the failure rate f_l for long trading positions and then to f_s , the failure rate for short trading positions.

In Table 3 we present complete VaR results (i.e. P-values for the Kupiec LR test) for the NASDAQ and NIKKEI stock indexes.¹⁶ In Table 4 we give summary results for the six series.

¹⁵In the literature on VaR models, this test is also called the Kupiec LR test, if the hypothesis is tested using a likelihood ratio test. See Kupiec (1995).

¹⁶Complete VaR results are available for all assets on request.

These results indicate that:

- VaR models based on the normal distribution (RiskMetrics and normal APARCH model) have a difficult job in modelling large positive and negative returns.
- the symmetric Student APARCH model improves considerably on the performance of normal based models but its performance is still not satisfactory in all cases. For the NASDAQ index, its performance in general is even worse than normal based models. The reason is that the critical values of the Student distribution $t_{\alpha,v}$ and $t_{1-\alpha,v}$ are very large in this case, which leads to a high level of long and short VaR: the model is often rejected because it is too conservative.¹⁷
- the skewed Student APARCH model improves on all other specifications for both negative and positive returns. For the NASDAQ the improvement is substantial as the switch to a skewed Student distribution alleviates almost all problems due to the ‘conservativeness’ of the symmetric Student distribution. The model performs correctly in 100% of all cases for the negative returns (long VaR) and for the positive returns (short VaR). As indicated in Table 4, the skewed Student APARCH model correctly models nearly all VaR levels for long and short positions. In all cases, this is a significant improvement on the VaR performances of symmetric models.

2.2.3 Out-of-sample VaR computation

The testing methodology in the previous subsection is equivalent to back-testing the model on the estimation sample. Therefore it can be argued that this should be favorable to the tested model. In a ‘real life situation’, VaR models are used to deliver out-of-sample forecasts, where the model is estimated on the known returns (up to time t for example) and the VaR forecast is made for period $[t + 1, t + h]$, where h is the time horizon of the forecasts. In this subsection we implement this testing procedure for the long and short VaR with $h = 1$ day.

We use an iterative procedure where the skewed Student APARCH model is estimated to predict the one-day-ahead VaR. The first estimation sample is the complete sample for which the data is available less the last five years. The predicted one-day-ahead VaR (both for long and short positions) is then compared with the observed return and both results are recorded for later assessment using the statistical tests. At the i -th iteration where i goes from 2 to $5 \cdot 252$ (five years of data), the estimation sample is augmented to include one more day and the VaRs are forecasted and recorded. Whenever i is a multiple of 50, the model is re-estimated to update the

¹⁷For example, the empirical failure rates for the short VaR are equal to 3.49%, 1.42%, 0.34%, 0.10% and 0.05% when α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%: in all cases the model is rejected because it is too conservative.

skewed Student APARCH parameters. In other words, we update the model parameters every 50 trading days and we thus assume a ‘stability window’ of 50 days for our parameters.¹⁸ We iterate the procedure until all days (less the last one) have been included in the estimation sample. Corresponding failure rates are then computed by comparing the long and short forecasted VaR_{t+1} with the observed return y_{t+1} for all days in the five years period. We use the same statistical tests as in the subsection dealing with the in-sample VaR.

Empirical results for the six financial assets are given in Table 5. Broadly speaking, these results are quite similar (although not as good) to those obtained for the in-sample testing procedure as the skewed Student APARCH model performs well for out-of-sample VaR prediction. Its combined (i.e. for long and short VaR) success rate is equal to 90% (FTSE), 70% (NASDAQ), 90% (NIKKEI), 80% (AA), 100% (MCD) and 80% (MRK). Moreover, there are no real differences between the results for the stock market indexes and the U.S. stocks as the skewed Student APARCH model performs equally well for both types of assets.

2.2.4 Expected short-fall and related measures

Our analysis in sub-sections 2.2.2 and 2.2.3 focused on the computation of empirical failure rates. In the last part of the empirical application, we now characterize the skewed Student APARCH model with respect to two other VaR related measures: the expected short-fall and the average multiple of tail event to risk measure.

The expected short-fall (see Scaillet, 2000) is defined as the expected value of the losses conditioned on the loss being larger than the VaR. The average multiple of tail event to risk measure “measures the degree to which events in the tail of the distribution typically exceed the VaR measure by calculating the average multiple of these outcomes to their corresponding VaR measures” (Hendricks, 1996). Both measures are computed for the in-sample estimation of the long and short VaR performed in sub-section 2.2.2.¹⁹

For the expected short-fall, we report full estimation results for the NASDAQ and NIKKEI stock indexes in Table 6.²⁰ These results indicate that the expected short-fall is in most cases larger (in absolute value) for the models based on the Student distribution than for the models based on the normal distribution. This is easily understood if one remembers that these models ‘fail’ less than the ones based on the normal distribution, but, when they fail, it happens for large (in

¹⁸In a previous version of the paper, we re-estimated the model each day, i.e. whenever a new observation entered the information set. This is extremely time consuming and the results are qualitatively the same as when one updates the model parameters every 5, 10 and 50 observations.

¹⁹The expected short-fall for the long VaR is computed as the average of the observed returns smaller than the long VaR. The expected short-fall for the short VaR is computed as the average of the observed returns larger than the short VaR. Computations are similar for the average multiple of tail event to risk measure.

²⁰Estimation results for the other 4 indexes are very similar to those given in Table 6 and are not reported.

absolute value) returns: the average of these returns is correspondingly large. It should be stressed that the expected short-fall is not a tool to rank VaR models or assess models' performances. Nevertheless it is useful for risk managers as it answers the following question: "when my model fails, how much do I lose on average?".

A related measure is the average multiple of tail event to risk measure, which is reported in Table 7 for the NASDAQ and NIKKEI stock indexes. The figures in this table indicate what is the average loss/predicted loss when the VaR model fails. For example, the 1.38 for the long VaR with NASDAQ data and the skewed Student APARCH models indicates that, at the 1% level, one expects to lose 1.38 the amount given by the VaR when returns are smaller than the VaR. As for the expected short-fall, this measure does not allow a ranking of the VaR models.

3 Multivariate VaR

Univariate empirical results given in the previous section have shown that the daily asset returns for the three stock indexes and the three U.S. stocks are heteroskedastic, fat-tailed and mostly skewed. Moreover, we have just shown that an AR-APARCH model combined with a skewed Student density for the innovations performs very well when the one-day-ahead VaR (both in- and out-of-sample) is to be forecasted.

In the second part of the paper, we now tackle the same in- and out-of-sample VaR problem but in a complete multivariate setting. As indicated in the introduction, most VaR studies focus on the univariate modelling of asset returns. From a finance point of view and whenever portfolio of assets are involved, the univariate approach has some severe shortcomings as one has to deal with the replicating portfolio (i.e. the univariate time series made up of the linear combination, with fixed weights, of the chosen assets) to get back to the univariate case and use the univariate VaR models. In this section, we extend the previous univariate analysis and we put forward a fully parameterized multivariate APARCH model (combined with a multivariate skewed Student distribution) to predict the long and short VaR for a portfolio of three assets.²¹ In addition to the flexibility of the multivariate approach, this model also accounts for the increasing correlation between asset returns during periods of high volatility.

3.1 A multivariate VaR model

More generally, let us consider a time series vector y_t , with $k(= 3)$ elements, $y_t = (y_{1t}, y_{2t}, \dots, y_{kt})'$. A multivariate dynamic regression model with time-varying means, variances and covariances for

²¹Our analysis can of course be extended to a portfolio of more than three assets. It is however sufficient to introduce a complete multivariate model without the additional burden of the rapidly increasing computation time.

the components of y_t can be written as:

$$y_t = \mu_t + \Sigma_t^{1/2} z_t \quad (12)$$

$$\mu_t = C(\mu|\Omega_{t-1}) \quad (13)$$

$$\Sigma_t = \Sigma(\mu, \eta|\Omega_{t-1}) \quad (14)$$

where $z_t \in \mathbb{R}^k$ is an *i.i.d.* random vector independent of Ω_{t-1} (the information set at time $t-1$) with zero mean and identity variance matrix and $C(\cdot|\Omega_{t-1})$ and $\Sigma(\cdot|\Omega_{t-1})$ are functions of Ω_{t-1} . It follows that $E(y_t|\mu, \Omega_{t-1}) = \mu_t$ and $Var(y_t|\mu, \eta, \Omega_{t-1}) = \Sigma_t^{1/2}(\Sigma_t^{1/2})' = \Sigma_t$, i.e. μ_t is the conditional mean vector (of dimension $k \times 1$) and Σ_t the conditional variance matrix (of dimension $k \times k$).

Among the most widely used multivariate GARCH models, one can highlight the Constant Conditional Correlations model (CCC) of Bollerslev (1990), the Vech of Kraft and Engle (1982) and Bollerslev, Engle and Wooldridge (1988), the BEKK of Engle and Kroner (1995), the Factor GARCH of Ng, Engle and Rothschild (1992), the General Dynamic Covariance (GDC) model of Kroner and Ng (1998), the Dynamic Conditional Correlations (DCC) model of Engle (2001) and the Time-Varying Correlation (TVC) model of Tse and Tsui (1998) (see Bauwens, Laurent and Rombouts, 2002 for a recent survey of multivariate GARCH models and their application in finance).

The specification we use to model the first two conditional moments is an obvious extension of the model introduced in the univariate part of the paper, i.e. an AR-TVC-APARCH model. This specification (which nests the TVC-GARCH proposed by Tse and Tsui, 1998) allows a time-varying conditional correlation, an APARCH specification for the conditional variances and an AR specification for the conditional mean. As suggested by the univariate results, we focus directly on an AR(2) and an APARCH(1,1) structure for the conditional means and conditional variances respectively. This AR(2)-TVC(1,1)-APARCH(1,1) model is defined as in Equation (12) with:

$$\mu_t = (\mu_{1,t}, \dots, \mu_{3,t})', \quad z_t = (z_{1,t}, \dots, z_{3,t})' \quad (15)$$

$$\mu_{i,t} = \mu_i + \sum_{j=1}^2 \phi_{i,j}(y_{i,t-j} - \mu_{i,j}) \quad (i = 1, \dots, 3) \quad (16)$$

$$\Sigma_t = D_t \Gamma_t D_t \quad (17)$$

$$D_t = \text{diag}(\sigma_{1,t}, \dots, \sigma_{3,t}) \quad (18)$$

$$\sigma_{i,t}^{\delta_i} = \omega_i + \alpha_{1,i} (|\varepsilon_{i,t-1}| - \alpha_{n,i} \varepsilon_{i,t-1})^{\delta_i} + \beta_i \sigma_{i,t-1}^2 \quad (i = 1, \dots, 3) \quad (19)$$

$$\varepsilon_t = (\varepsilon_{1,t}, \dots, \varepsilon_{3,t})' = y_t - \mu_t \quad (20)$$

$$\Gamma_t = (1 - \theta_1 - \theta_2)\Gamma + \theta_1 \Gamma_{t-1} + \theta_2 \Psi_{t-1} \quad (21)$$

$$\Gamma = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix} \quad (22)$$

$$\Psi_{t-1} = B_{t-1}^{-1} E_{t-1} E_{t-1}' B_{t-1}^{-1} \quad (23)$$

$$B_{t-1} = \text{diag} \left(\sum_{h=1}^m \epsilon_{1,t-h}^2, \dots, \sum_{h=1}^m \epsilon_{3,t-h}^2 \right)^{1/2} \quad (24)$$

$$E_{t-1} = (\epsilon_{t-1}, \dots, \epsilon_{t-m}) \quad (25)$$

$$\epsilon_t = (\epsilon_{1,t}, \dots, \epsilon_{3,t})' = D_t^{-1} \varepsilon_t \quad (26)$$

where $\mu_i, \phi_{i,1}, \phi_{i,2}, \omega_i, \alpha_i, \alpha_{n,i}, \delta_i, \beta_i$ ($i = 1, \dots, 3$), ρ_{ij} ($1 \leq i < j \leq 3$), and θ_1, θ_2 are parameters to be estimated.²² Ψ_{t-1} is thus the sample correlation matrix of $\{\epsilon_{t-1}, \dots, \epsilon_{t-m}\}$. Since $\Psi_{t-1} = 1$ if $m = 1$, we must take $m \geq 3$ to have a non-trivial correlation. In this application, we set $m = 3$. Note that the TVC-MGARCH model nests the constant correlation GARCH model of Bollerslev (1990). Therefore, we can test $\theta_1 = \theta_2 = 0$ to check whether the constant correlation assumption is appropriate.

Estimation of the AR-TVC-APARCH model is done by (approximate) maximum likelihood. Thus, one has to make an additional assumption on the innovation process. As for the univariate case, two natural candidates are the multivariate normal and the multivariate Student density with at least two degrees of freedom v (in order to ensure the existence of second moments). The second distribution, denoted $z \sim ST(0, I_k, v)$, may be defined as follows:

$$g(z_t|v) = \frac{\Gamma\left(\frac{v+k}{2}\right)}{\Gamma\left(\frac{v}{2}\right) [\pi(v-2)]^{\frac{k}{2}}} \left[1 + \frac{z_t' z_t}{v-2} \right]^{-\frac{k+v}{2}}, \quad (27)$$

where $\Gamma(\cdot)$ is the Gamma function.

Let us consider now the portfolio $p_t^* = W' y_t$, where W is the vector of the percentage of wealth invested in the k assets. Accordingly, under the normal assumption, the one-step-ahead VaR computed in $t-1$ for long trading positions is given by $W' \mu_t + \sqrt{W' \Sigma_t W} z_\alpha$, while for short trading positions it is equal to $W' \mu_t + \sqrt{W' \Sigma_t W} z_{1-\alpha}$, with z_α being the left quantile at $\alpha\%$ for the normal distribution and $z_{1-\alpha}$ is the right quantile at $\alpha\%$. Under the assumption of multivariate Student innovations, the one-step-ahead VaR is obtained by replacing z_α and $z_{1-\alpha}$ by $st_{\alpha,v}$ and $st_{1-\alpha,v}$, respectively.

However, and as shown in Section 2, these densities are known to give poor VaR forecasts when the returns are skewed, even when an asymmetric specification for the conditional variance is included in the model. For this reason, Bauwens and Laurent (2002) recently put forward a practical and flexible solution to introduce skewness in multivariate symmetrical distributions. An application of this procedure to the multivariate Student density in Equation (27) gives a

²²The parameters θ_1 and θ_2 are assumed to be nonnegative with the additional constraint that $\theta_1 + \theta_2 < 1$.

‘multivariate skewed Student’ density, for which each marginal is a univariate skewed Student as described by Equation (6) and has its own asymmetry coefficient (ξ_i).

From Definition 2 of Bauwens and Laurent (2002), the multivariate skewed Student density, denoted $z \sim SKST(0, I_k, \xi, \nu)$, is given by:

$$f(z|\xi, \nu) = \left(\frac{2}{\sqrt{\pi}}\right)^k \left(\prod_{i=1}^k \frac{\xi_i s_i}{1 + \xi_i^2}\right) \frac{\Gamma(\frac{\nu+k}{2})}{\Gamma(\frac{\nu}{2})(\nu-2)^{\frac{k}{2}}} \left(1 + \frac{\kappa' \kappa}{\nu-2}\right)^{-\frac{k+\nu}{2}}. \quad (28)$$

where

$$\xi = (\xi_1, \dots, \xi_k) \quad (29)$$

$$\kappa = (\kappa_1, \dots, \kappa_k)' \quad (30)$$

$$\kappa_i = (s_i z_i + m_i) \xi_i^{-I_i} \quad (31)$$

$$I_i = \begin{cases} 1 & \text{if } z_i \geq -\frac{m_i}{s_i} \\ -1 & \text{if } z_i < -\frac{m_i}{s_i}. \end{cases}$$

Note that m_i and s_i are obtained by replacing ξ by ξ_i in Equations (7) and (8).

By construction, $E(z) = 0$ and $Var(z) = I_k$. If $\xi_i = 1$ for all i , the $SKST(0, I_k, \xi, \nu)$ density is the $ST(0, I_k, \nu)$ one, i.e. the symmetric Student density and thus it also nests the multivariate normal if furthermore $1/\nu = 0$.

Under the assumption that $z \sim SKST(0, I_k, \xi, \nu)$, the computation of the one-step-ahead VaR is not as straightforward as before. Indeed there are no ‘direct’ analytic and easy-to-use formulae to switch from the conditional volatilities to the long and short VaR of the portfolio. However, the VaR can be computed using a simple Monte Carlo simulation as widely used in quantitative finance and option pricing. Indeed, let us compute a set of possible one-day-ahead prices for the chosen portfolio $p_j^* = W' \mu_t + \sqrt{W' \Sigma_t W} z_j(\xi, \nu)$, for $j = 1, \dots, j^*$, where $z_j(\xi, \nu)$ is a simulated random variable with distribution $SKST(0, I_k, \xi, \nu)$.²³ By definition, the one-step-ahead VaR at $\alpha\%$ is defined as the empirical quantile at $\alpha\%$ of p_j^* (over the j^* simulations), see Jorion (2000) for a discussion of Monte Carlo techniques in VaR applications. In the empirical application given in the next sub-section, we set $j^* = 100,000$.²⁴

3.2 In-sample and out-of-sample VaR computation in a multivariate setting

As detailed at the start of Section 3, we consider a portfolio of three assets which are the three U.S. stocks we dealt with in the univariate VaR application. We suppose that an investor has an amount of \$1 of which he invests 70% in AA, 20% in MCD and 10% in MRK.²⁵

²³See Bauwens and Laurent (2002) for more details on how to generate $SKST(0, I_k, \xi, \nu)$ random numbers.

²⁴A separate Monte Carlo simulation study, not reported here, has shown that a choice of 100,000 simulations provides accurate estimates of the quantile.

²⁵We choose these weights to favor the asset which has a large skewness to fully test our method.

Estimation results for the multivariate skewed Student AR-TVC-APARCH, obtained using a preliminary version of the Ox package MG@RCH 1.0 (see Laurent, Peters and Rombouts, 2002), are given in Table 8. Note that θ_1 and θ_2 are individually significantly different from 0 and the joint hypothesis $H_0 : \theta_1 = \theta_2 = 0$ is widely rejected as the LR test statistic is equal to 53.95, much larger than the critical value of the corresponding $\chi^2(2)$. Therefore (and it was widely expected), the constant correlation hypothesis is largely rejected.

In-sample VaR results for long (i.e. a portfolio for an investor who is long in 70% of AA, 20% of MCD and 10% of MRK) and short (i.e. a portfolio for an investor who is short in 70% of AA, 20% of MCD and 10% of MRK) trading positions are reported in Table 9 for the normal, Student and skewed Student AR-TVC-APARCH models. In addition to the ML estimation of the models, we use the Monte Carlo simulation method to compute the left and right quantiles (long and short VaR) at the required percentage level. The empirical evidence is once again very much in favor of the skewed Student ARCH class of models, but this time in a multivariate setting. Indeed, the multivariate skewed Student AR-TVC-APARCH models exhibits failure rates for the portfolio which are very close to their theoretical counterparts as the P-values are all larger than the conventional levels of significance. For the other two models, the evidence is rather mixed, with the Student version delivering acceptable results and the normal-based model underperforming. Finally we also report (bottom lines of both panels of Table 9) full out-of-sample VaR results for the skewed Student AR-TVC-APARCH model. As in the univariate application, we use a five-year out-of-sample period for the out-of-sample assessment, with the model being re-estimated every 50 observations. Once again the performance of the skewed Student-based model is excellent as all P-values are larger than 5%. The total success rate (i.e. for both long and short VaR and for all percentage levels) of this model is thus equal to 100%, in-sample or out-of-sample.

4 Conclusion

Over short-term time horizons, conditional VaR models are usually found to be good candidates for quantifying possible trading losses. In this paper, we extended this analysis by introducing both univariate and multivariate VaR models that take into account losses arising from long and short trading positions. Because of the nature of long and short trading, this translates into bringing forward statistical models that correctly model the left and right tails of the distribution of returns. The proposed models are the skewed Student APARCH model (in the univariate setting) and the skewed Student AR-TVC-APARCH model (in the multivariate analysis suited for portfolio applications). Because density distribution of returns are usually not symmetric, it is shown that models²⁶ that rely on symmetric normal or Student distributions underperform

²⁶We considered three symmetric volatility models: the RiskMetrics, normal and Student APARCH models.

with respect to the new models when the one-day-ahead VaR is to be forecasted. All models were applied to daily data for three stock indexes (FTSE, NASDAQ and NIKKEI) and three U.S. stocks (Alcoa, McDonald and Merck), with an out-of-sample testing procedure confirming the results of the in-sample backtesting method: in all cases the skewed Student-based models performed very well, both in the univariate and multivariate settings.

At this stage, several extensions can be considered. First, the performance of the new VaR models could also be assessed on multi-day period forecasts. While VaR models based on ARCH class specifications perform rather well for one-day time horizons, it is known that their performance is not as good for long time periods. Some recent work in this field is Christoffersen and Diebold (2000). Secondly, the VaR for long and short trading positions could be computed using non-parametric VaR models. Computation times and quality of VaR forecasts could be compared with the results given by the skewed Student APARCH class of models. Thirdly, as argued recently by Engle and Patton (1999), time-varying higher conditional moments are clearly of interest. In this respect, Hansen (1994), Harvey and Siddique (1999) and Lambert and Laurent (2000) have had some success in introducing dynamics in the third and fourth moments. Finally, additional relevant variables could be included in the conditional variance equations of the models. For example and given a database with this kind of information is available, one could assess the information content of the lagged implied volatility (computed from short-term call and put options written on the underlying assets) regarding the volatility or VaR forecasts. Previous research work in this field include Day and Lewis (1992) (options on the S&P100 index), Xu and Taylor (1995) (PHLX currency options market) or more recently Giot (2002) (options on nearby future prices for the cocoa, coffee and sugar contracts traded on the New York Board of Trade).

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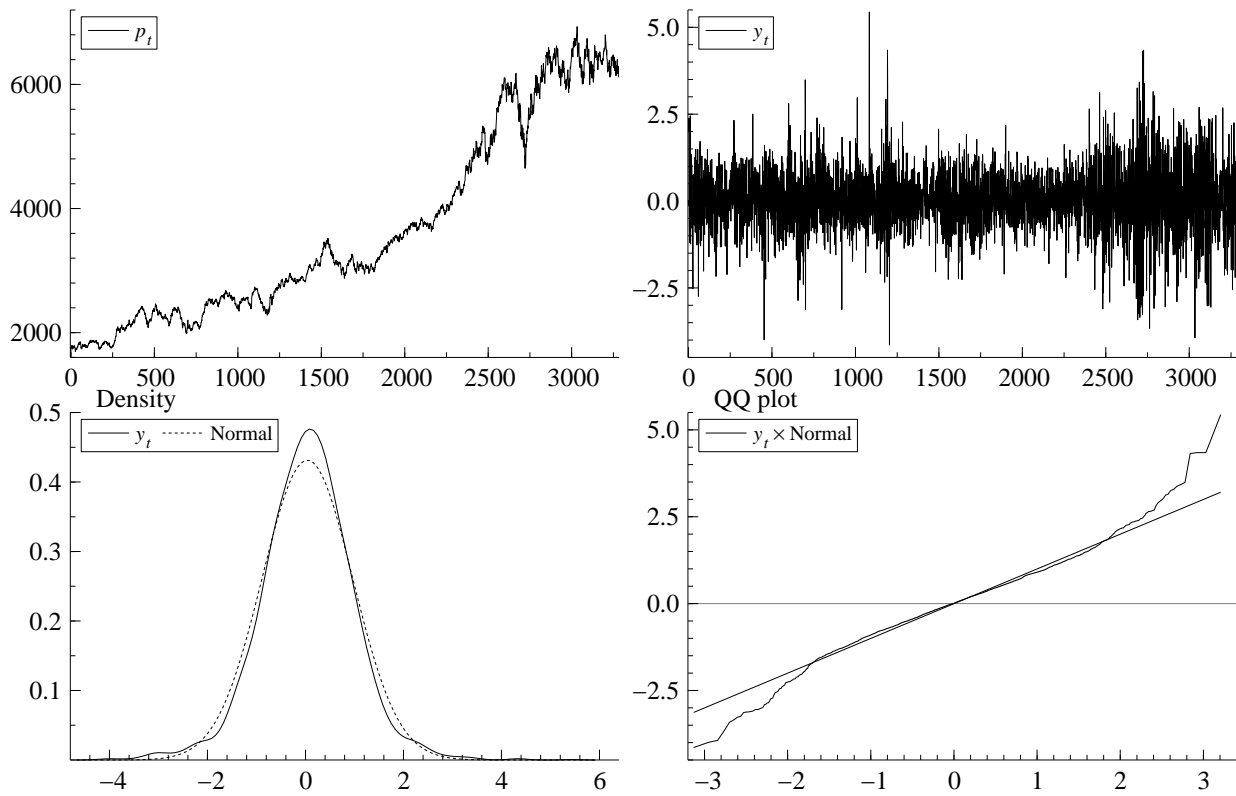


Figure 1: FTSE stock index in level, daily returns, daily returns density (vs. normal) and QQ-plot against the normal distribution. The time period is 04/01/1988 - 21/12/2000.

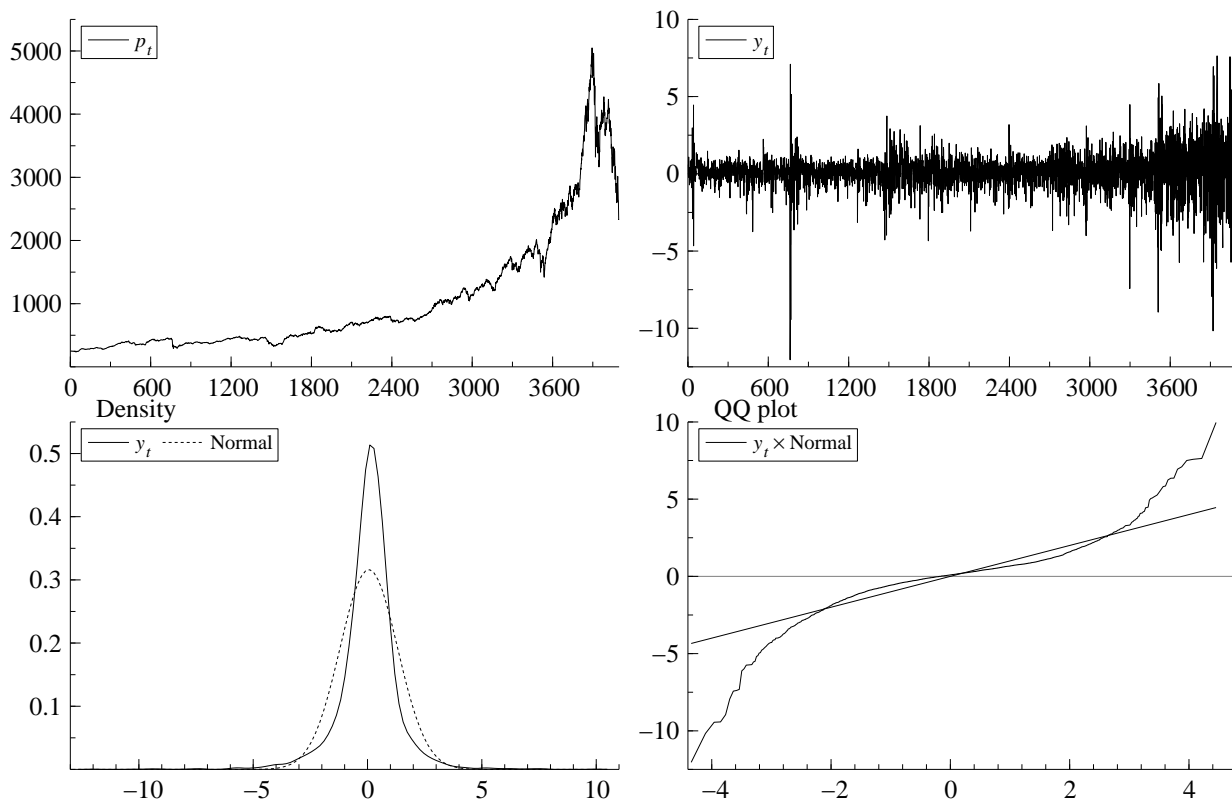


Figure 2: NASDAQ stock index in level, daily returns, daily returns density (vs. normal) and QQ-plot against the normal distribution. The time period is 11/10/1984 - 21/12/2000.

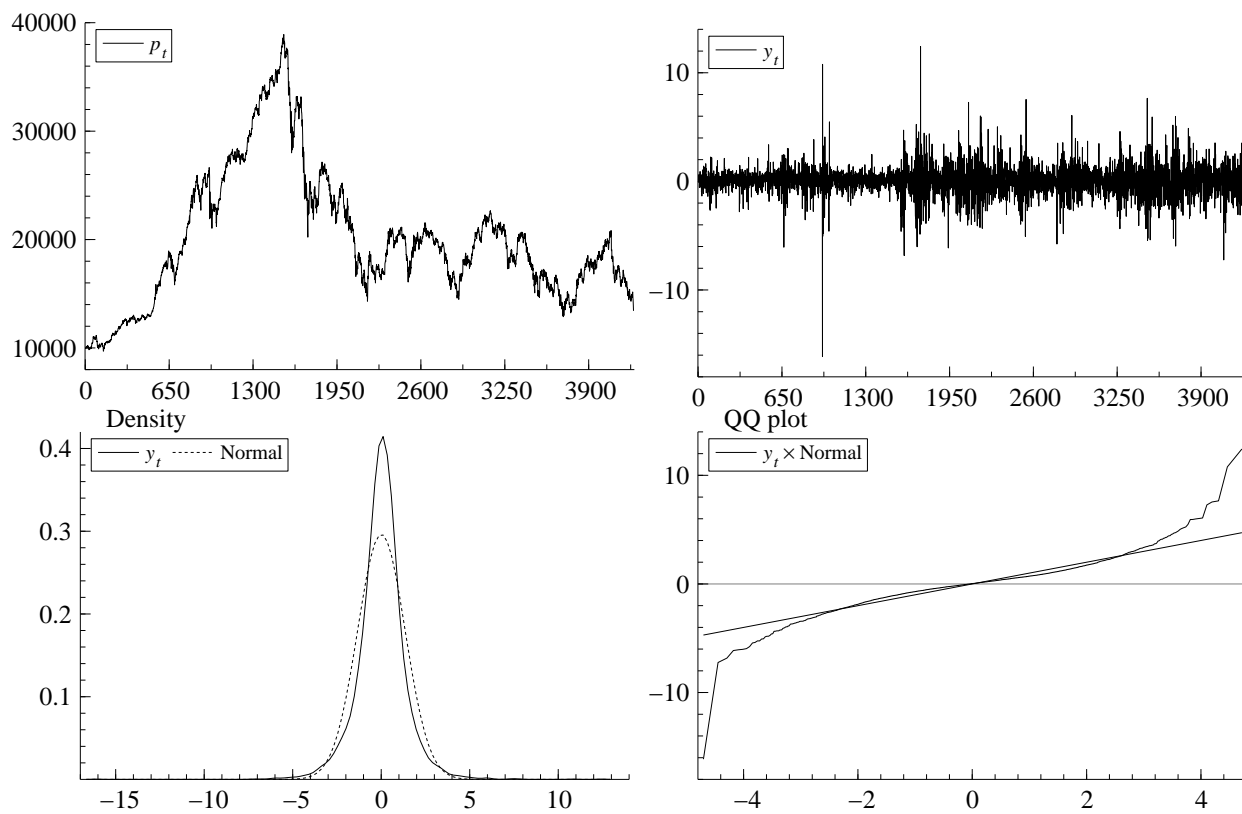


Figure 3: NIKKEI stock index in level, daily returns, daily returns density (vs. normal) and QQ-plot against the normal distribution. The time period is 4/1/1984 - 21/12/2000.

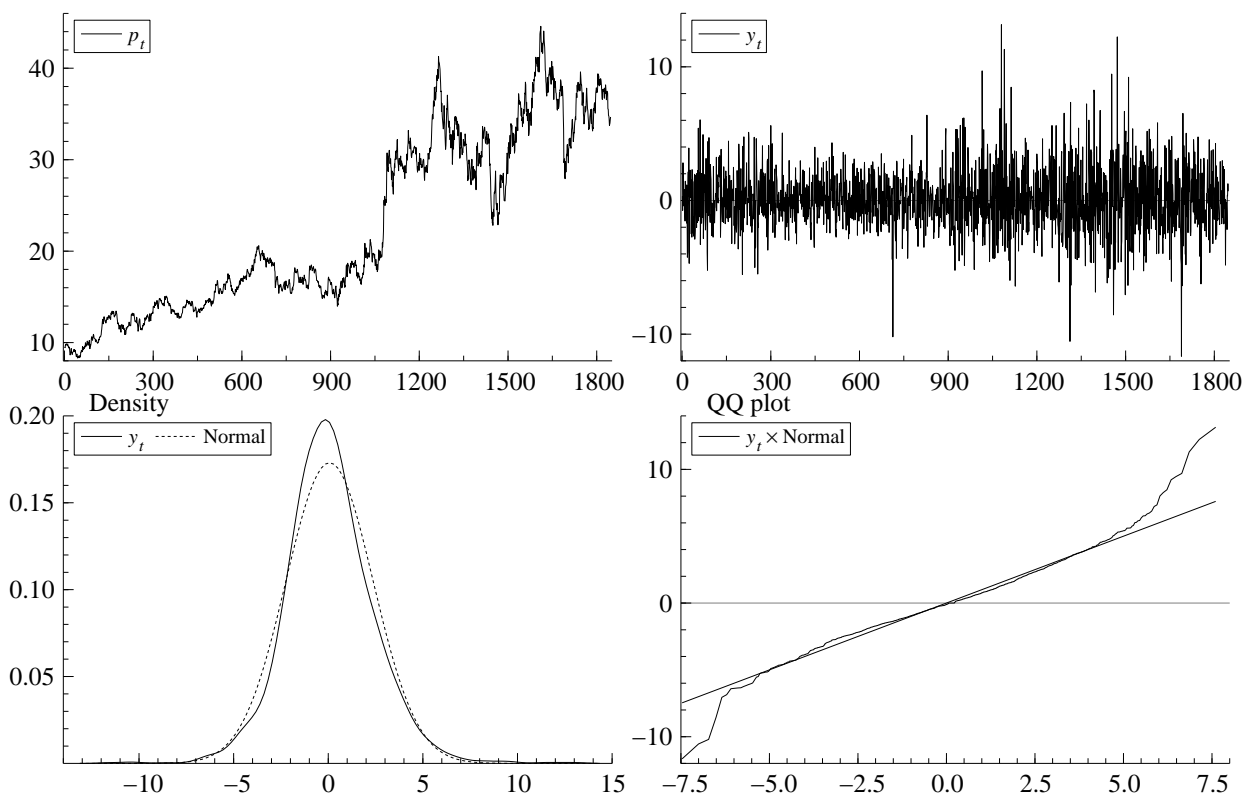


Figure 4: ALCOA in level, daily returns, daily returns density (vs. normal) and QQ-plot against the normal distribution. The time period is 03/01/1990 - 03/05/2002.

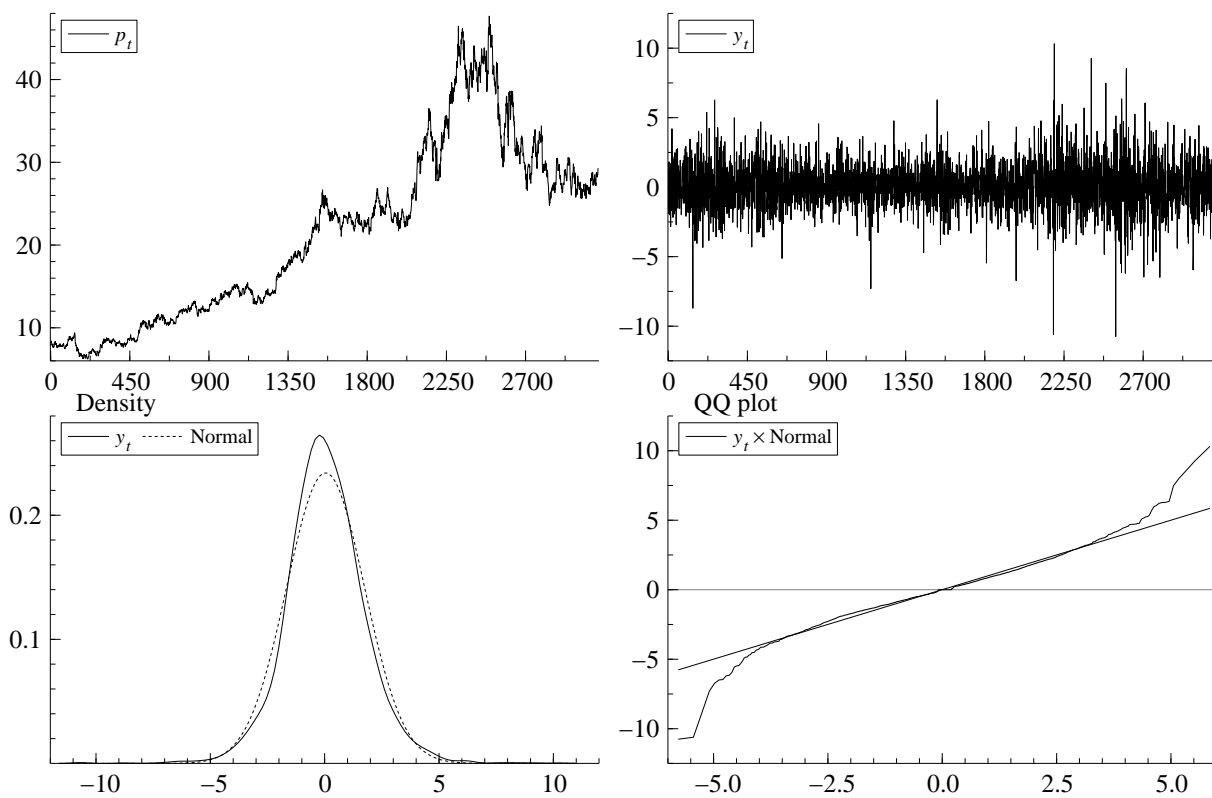


Figure 5: MacDonalD in level, daily returns, daily returns density (vs. normal) and QQ-plot against the normal distribution. The time period is 03/01/1990 - 03/05/2002.

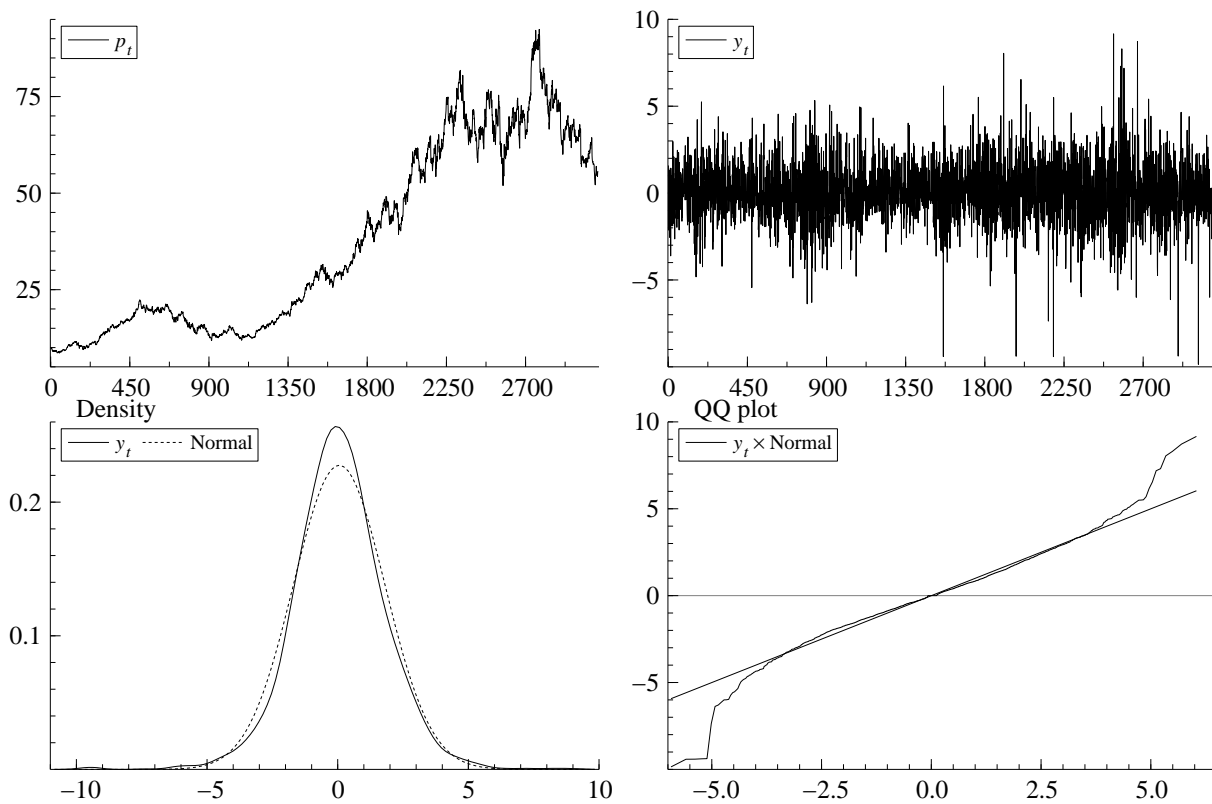


Figure 6: Merck in level, daily returns, daily returns density (vs. normal) and QQ-plot against the normal distribution. The time period is 03/01/1990 - 03/05/2002.

Table 1: Descriptive statistics

	Stock indexes			Stocks		
	FTSE	NASDAQ	NIKKEI	AA	MCD	MRK
Annual s.d.	14.68	20.03	21.38	32.31	27.04	27.83
Skewness	-0.04	-0.74	-0.14	0.30	0.06	-0.08
Excess Kurtosis	1.82	11.25	10.15	3.07	2.88	2.50
Minimum	-4.14	-12.04	-16.14	-11.66	-10.76	-9.85
Maximum	5.443	9.96	12.43	13.16	10.31	9.16
$Q^2(10)$	621.1	2874.4	590.6	284.9	233.8	156.9

Descriptive statistics for the daily returns of the corresponding financial asset (stock index or individual stock) expressed in %. All values are computed using PcGive. $Q^2(10)$ is the Ljung-Box Q-statistic of order 10 on the squared series.

Table 2: Skewed Student APARCH

	Stock indexes			Stocks		
	FTSE	NASDAQ	NIKKEI	AA	MCD	MRK
ω	0.007 (0.003)	0.015 (0.004)	0.024 (0.004)	0.012 (0.006)	0.016 (0.008)	0.042 (0.014)
α_1	0.042 (0.008)	0.126 (0.014)	0.105 (0.011)	0.039 (0.009)	0.026 (0.008)	0.049 (0.010)
α_n	0.365 (0.111)	0.278 (0.057)	0.493 (0.071)	0.293 (0.130)	0.089 (0.101)	0.586 (0.147)
β_1	0.955 (0.008)	0.889 (0.014)	0.897 (0.010)	0.964 (0.009)	0.970 (0.007)	0.937 (0.013)
δ	1.416 (0.247)	1.104 (0.149)	1.168 (0.134)	1.052 (0.231)	1.793 (0.365)	1.022 (0.188)
$\log(\xi)$	-0.060 (0.027)	-0.184 (0.022)	-0.054 (0.022)	0.096 (0.026)	0.088 (0.026)	0.047 (0.026)
ν	12.783 (2.265)	6.694 (0.664)	6.511 (0.590)	7.946 (1.027)	7.643 (0.924)	7.411 (0.861)
V	0.994	1.003	0.985	0.992	0.993	0.973
$Q^2(10)$	7.76	16.97	13.20	15.72	41.81	5.76

Estimation results for the volatility specification of the skewed Student APARCH model. Standard errors are reported in parentheses. $V = \alpha_1 E(|z| - \alpha_n z)^\delta + \beta_1$ while $Q^2(10)$ is the Ljung-Box Q-statistic of order 10 computed on the squared standardized residuals.

Table 3: VaR results for NASDAQ and NIKKEI (in-sample)

α	5%	2.5%	1%	0.5%	0.25%
VaR for long positions (NASDAQ)					
RiskMetrics	0.001	0	0	0	0
N APARCH	0.115	0	0	0	0
ST APARCH	0	0	0.050	0.075	0.053
SKST APARCH	0.600	0.738	0.641	0.918	0.406
VaR for long positions (NIKKEI)					
RiskMetrics	0	0	0	0	0
N APARCH	0.422	0.294	0.003	0.047	0
ST APARCH	0.087	0.569	0.588	0.787	0.479
SKST APARCH	0.590	0.832	0.017	0.470	0.677
VaR for short positions (NASDAQ)					
RiskMetrics	0.001	0.015	0.991	0.331	0.264
N APARCH	0	0.001	0.031	0.303	0.812
ST APARCH	0	0	0	0	0.002
SKST APARCH	0.506	0.894	0.429	0.303	0.151
VaR for short positions (NIKKEI)					
RiskMetrics	0.005	0.988	0.016	0	0
N APARCH	0	0.031	0.325	0.006	0
ST APARCH	0.002	0.024	0.484	0.425	0.906
SKST APARCH	0.069	0.223	0.590	0.316	0.205

P-values for the null hypothesis $f_l = \alpha$ (i.e. failure rate for the long trading positions is equal to α , top of the table) and $f_s = \alpha$ (i.e. failure rate for the short trading positions is equal to α , bottom of the table). α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%. The models are successively the RiskMetrics, normal APARCH, Student APARCH and skewed Student APARCH models.

Table 4: VaR results for all stock indexes and individual stocks (in-sample)

VaR for long positions						
	Stock indexes			Stocks		
	FTSE	NASDAQ	NIKKEI	AA	MCD	MRK
RiskMetrics	40	0	0	40	60	40
N APARCH	60	20	40	100	60	40
ST APARCH	100	60	100	80	80	80
SKST APARCH	100	100	80	100	100	100

VaR for short positions						
	Stock indexes			Stocks		
	FTSE	NASDAQ	NIKKEI	AA	MCD	MRK
RiskMetrics	60	60	20	20	20	20
N APARCH	40	40	20	20	20	80
ST APARCH	80	0	60	100	100	100
SKST APARCH	100	100	100	100	100	60

Number of times (out of 100) that the null hypothesis $f_l = \alpha$ (i.e. failure rate for the long trading positions is equal to α , top of the table) is not rejected and $f_s = \alpha$ (i.e. failure rate for the short trading positions is equal to α , bottom of the table) is not rejected for the combined five possible values of α (the level of significance is 5%). The models are successively the RiskMetrics, normal APARCH, Student APARCH and skewed Student APARCH models.

Table 5: VaR results
(Skewed Student APARCH, out-of-sample)

α	5%	2.5%	1%	0.5%	0.25%
VaR for long positions					
FTSE	0.025	0.192	0.151	0.515	0.645
NASDAQ	0.003	0.071	0.509	0.514	0.645
NIKKEI	0.035	0.427	0.643	0.324	0.156
AA	0.034	0.033	0.237	0.515	0.377
MCD	0.166	0.192	0.092	0.090	0.153
MRK	0.207	0.334	0.053	0.003	0.022
VaR for short positions					
FTSE	0.166	0.405	0.282	0.904	0.932
NASDAQ	0	0	0.355	0.783	0.645
NIKKEI	0.609	0.785	0.237	0.515	0.932
AA	0.897	0.334	0.237	0.590	0.337
MCD	0.513	0.656	0.910	0.590	0.932
MRK	0.233	0.405	0.864	0.904	0.337

P-values for the null hypothesis $f_t = \alpha$ (i.e. failure rate for the long trading positions is equal to α , top of the table) and $f_s = \alpha$ (i.e. failure rate for the short trading positions is equal to α , bottom of the table). α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%. The failure rates are computed for the skewed Student APARCH model (out-of-sample estimation).

Table 6: Expected short-fall for NASDAQ and NIKKEI
(in-sample)

α	5%	2.5%	1%	0.5%	0.25%
Expected short-fall for long positions (NASDAQ)					
RiskMetrics	-2.26	-2.43	-2.73	-2.87	-3.11
N APARCH	-2.41	-2.60	-2.97	-3.39	-3.61
ST APARCH	-2.28	-2.57	-3.19	-3.73	-4.25
SKST APARCH	-2.42	-2.75	-3.63	-4.04	-4.38
Expected short-fall for long positions (NIKKEI)					
RiskMetrics	-2.50	-2.79	-3.17	-3.45	-3.51
N APARCH	-2.65	-3.07	-3.56	-3.98	-4.24
ST APARCH	-2.59	-3.10	-3.83	-4.44	-5.23
SKST APARCH	-2.65	-3.20	-4.16	-4.58	-5.46
Expected short-fall for short positions (NASDAQ)					
RiskMetrics	2.20	2.64	2.77	2.99	3.46
N APARCH	2.35	2.56	3.04	3.27	3.30
ST APARCH	2.33	2.65	3.41	3.02	3.72
SKST APARCH	2.08	2.51	2.80	3.31	2.57
Expected short-fall for short positions (NIKKEI)					
RiskMetrics	2.75	3.12	3.99	4.20	4.32
N APARCH	2.82	3.37	3.84	4.13	4.30
ST APARCH	2.78	3.48	4.39	4.42	5.08
SKST APARCH	2.69	3.33	4.10	4.48	4.67

Expected short-fall (in-sample evaluation) for the long and short VaR (at level α) given by the normal APARCH, Student APARCH, RiskMetrics and skewed Student APARCH models. α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%.

Table 7: Average multiple of tail event to risk measure
for NASDAQ and NIKKEI (in-sample)

α	5%	2.5%	1%	0.5%	0.25%
AMTERM for long positions (NASDAQ)					
RiskMetrics	1.52	1.46	1.42	1.39	1.39
N APARCH	1.44	1.37	1.38	1.41	1.37
ST APARCH	1.48	1.38	1.41	1.42	1.42
SKST APARCH	1.43	1.38	1.39	1.40	1.36
AMTERM for long positions (NIKKEI)					
RiskMetrics	1.47	1.40	1.37	1.42	1.40
N APARCH	1.40	1.36	1.33	1.46	1.43
ST APARCH	1.40	1.37	1.41	1.49	1.50
SKST APARCH	1.39	1.35	1.51	1.49	1.48
AMTERM for short positions (NASDAQ)					
RiskMetrics	1.30	1.28	1.22	1.20	1.21
N APARCH	1.26	1.22	1.20	1.19	1.16
ST APARCH	1.27	1.21	1.19	1.26	1.26
SKST APARCH	1.28	1.23	1.20	1.18	1.22
AMTERM for short positions (NIKKEI)					
RiskMetrics	1.38	1.34	1.30	1.30	1.30
N APARCH	1.36	1.35	1.31	1.27	1.23
ST APARCH	1.38	1.34	1.28	1.19	1.16
SKST APARCH	1.39	1.35	1.28	1.23	1.16

Average multiple of tail event to risk measure (AMTERM, in-sample evaluation) for the long and short VaR (at level α) given by the normal APARCH, Student APARCH, RiskMetrics and skewed Student APARCH models. α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%.

Table 8: Multivariate skewed Student AR-TVC-APARCH

	AA	MCD	MRK
θ_1		0.9915 (0.001)	
θ_2		0.0078 (0.001)	
ρ_{ij}			
AA	1	0.4207 (0.111)	0.4506 (0.177)
MCD		1	0.5403 (0.123)
MRK			1
$\log(\xi_i)$	0.1050 (0.025)	0.0903 (0.027)	0.0566 (0.029)
v		8.5334 (0.678)	

Estimation results for the volatility specification of the multivariate skewed Student AR-TVC-APARCH model. Standard errors are reported in parentheses.

Table 9: Portfolio VaR, with the AR-TVC-APARCH model

α	5%	2.5%	1%	0.5%	0.25%
VaR for long positions					
In-sample					
Normal	0.007	0.131	0.567	0.911	0.045
Student	0.007	0.033	0.032	0.503	0.775
Skewed Student	0.961	0.362	0.567	0.911	0.445
Out-of-sample					
Skewed Student	0.445	0.334	0.910	0.783	0.337
VaR for short positions					
In-sample					
Normal	0.830	0.090	0.001	0.002	0.010
Student	0.493	0.300	0.391	0.886	0.775
Skewed Student	0.192	0.303	0.125	0.070	0.505
Out-of-sample					
Skewed Student	0.897	0.929	0.697	0.515	0.645

P-values for the null hypothesis $f_l = \alpha$ (i.e. failure rate for the long trading positions is equal to α , top of the table) and $f_s = \alpha$ (i.e. failure rate for the short trading positions is equal to α , bottom of the table). α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%. The models are successively the normal, Student and skewed Student AR-TVC-APARCH models.